Fundamentals of Solid State Physics

Electronic Properties - The Free Electron Model

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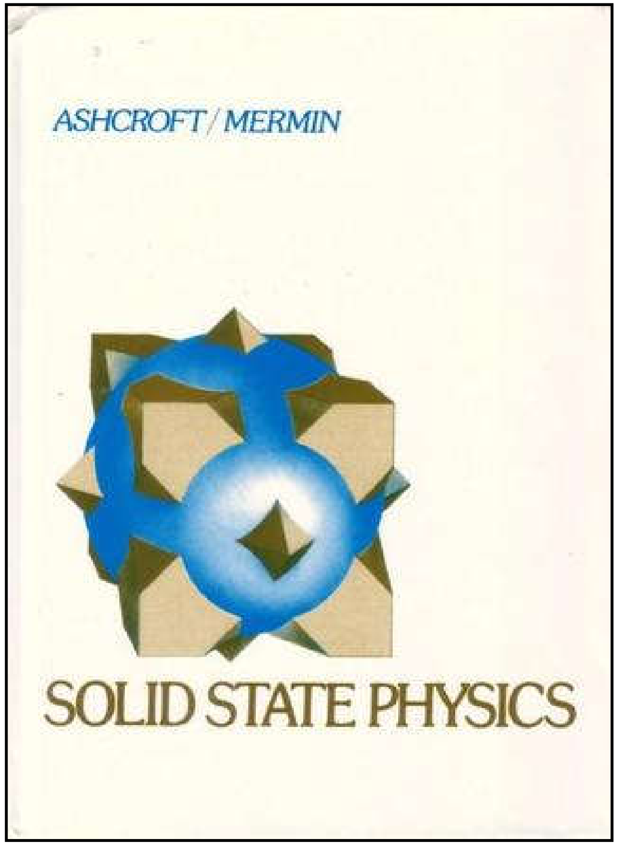
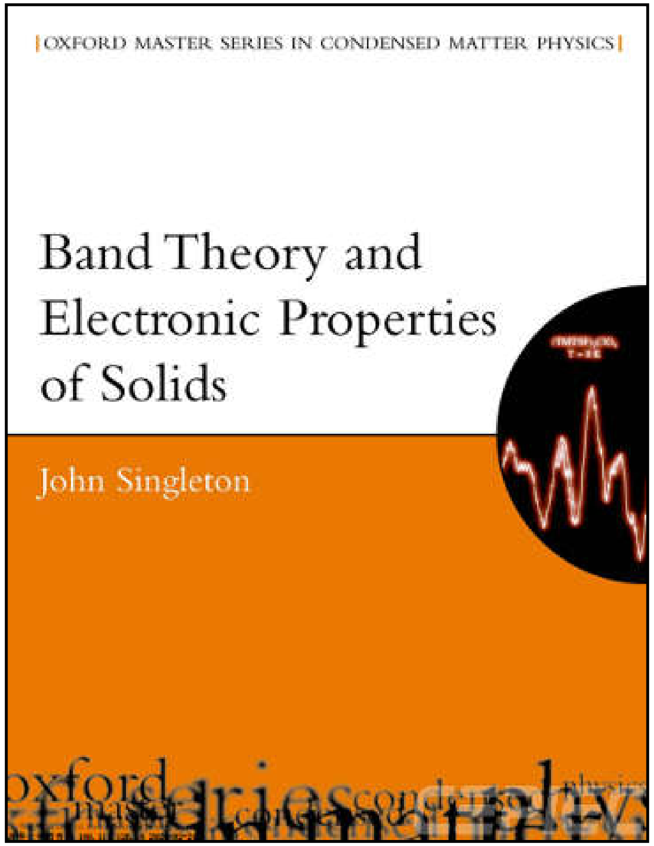
This Class

- Introduction (Week 1)
- Materials and Crystal Structures (Week 2–3)
- **Electronic Properties (Week 4–12)**
  - Free electrons (the Drude and Sommerfeld models)
  - Electrons in a periodic potential (Bloch's Theorem)
  - The near-free electron model, the tight-binding model
  - Electronic band diagram, band gaps, effective mass
  - Metals, insulators, semiconductors
  - Devices: junctions, diodes, transistors
- Thermal Properties (Week 13)
- Optical Properties (Week 14)
- Magnetic Properties (Week 15)

Further Reading

- **Ashcroft & Mermin, Chapter 1, 2, 3**
- **Singleton, Chapter 1**

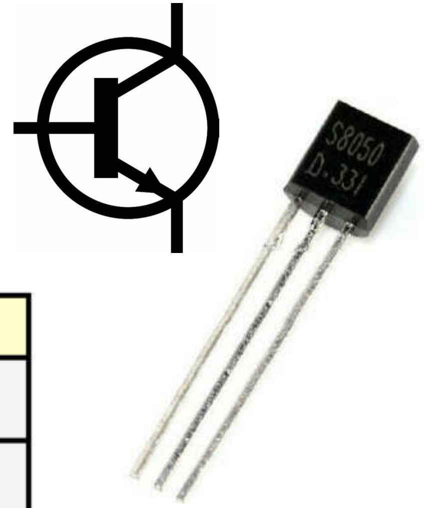




Electronic Properties of Materials

CMOS transistor

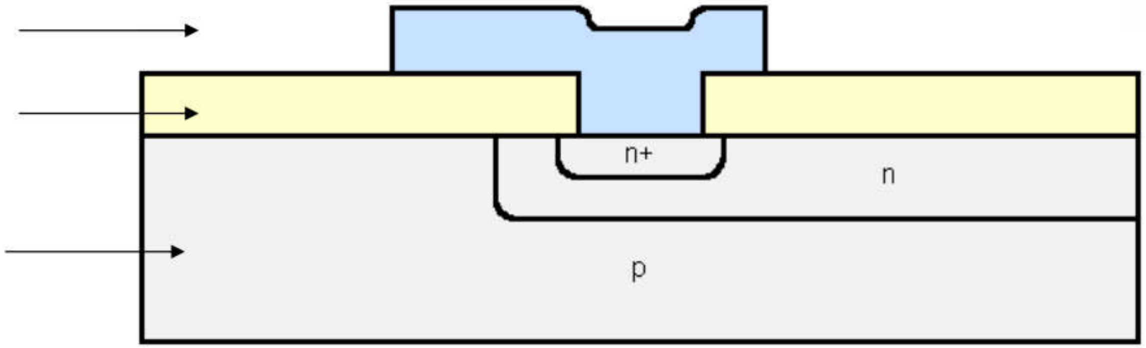
- Complementary Metal-Oxide-Semiconductor



conductor Al

insulator SiO_2

semiconductor Si



Metal

SiO_2

Silicon

Electronic Properties of Metals

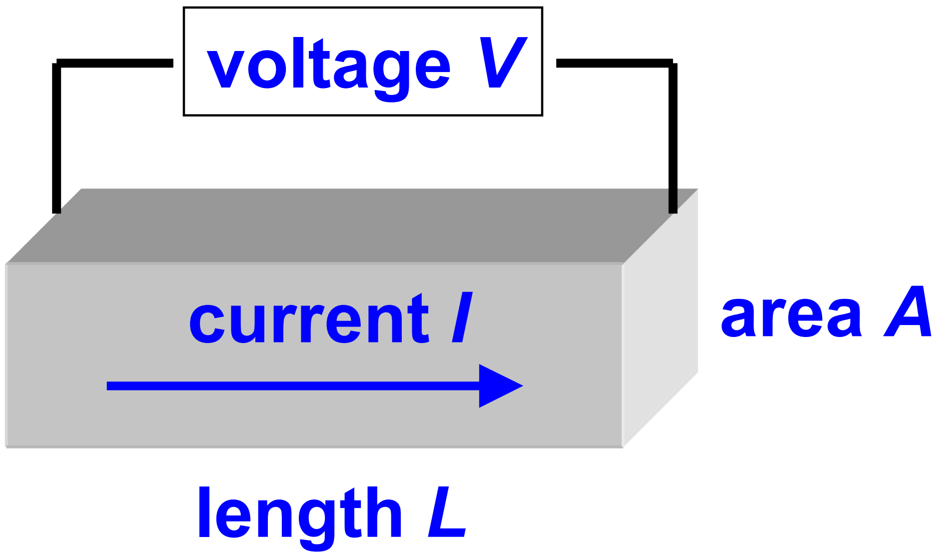
- **Ohm's Law (1827)**

$$V = IR$$

macroscopic

resistance

$$R = \rho \frac{L}{A}$$



$$j = \frac{I}{A} = \frac{V}{AR} = \frac{EL}{A\rho L / A} = \frac{E}{\rho} = \sigma E$$

microscopic

j **- current density (A/m^2)**
E **- electric field (V/m)**
ρ **- resistivity (Ω*m)**
σ **- conductivity (S/m)**

Electronic Properties of Metals

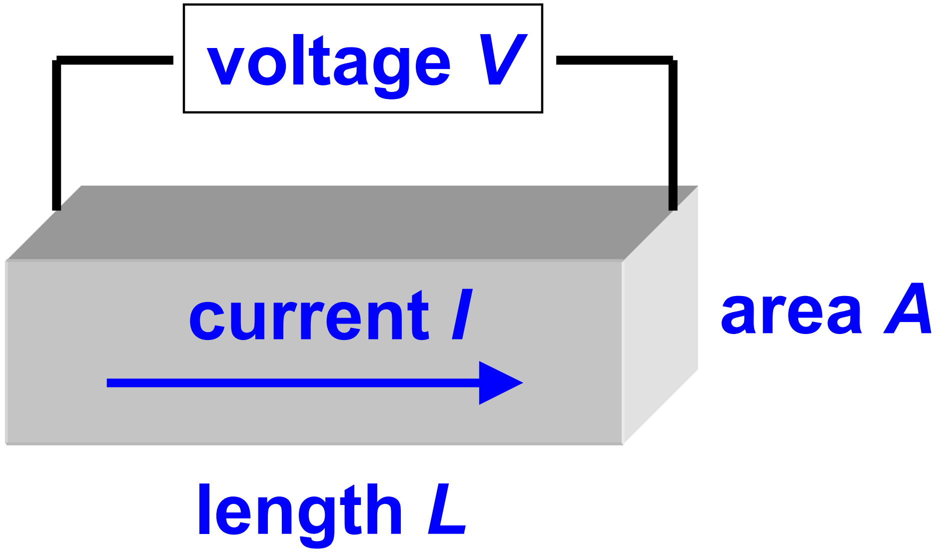
- **conductivity and mobility**

$$I = \frac{\partial Q}{\partial t} = ne\frac{\partial V}{\partial t} = neA\frac{\partial L}{\partial t} = neAv$$

$$j = \frac{I}{A} = nev$$

$$\sigma = \frac{j}{E} = ne\frac{v}{E} = ne\mu$$

$$\mu = \frac{v}{E}$$



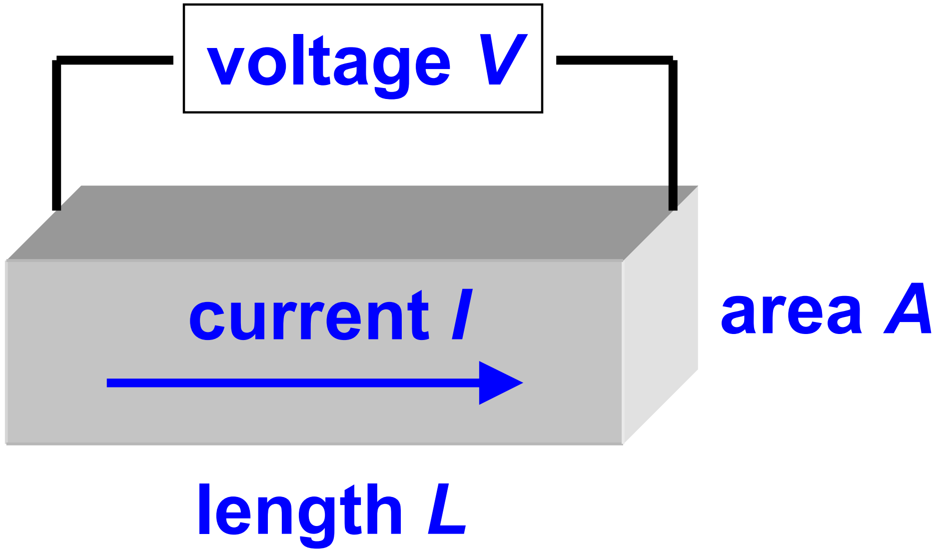
n **- density of electrons (#/m³)**

v **- velocity of electrons (m/s)**

μ **- electron mobility (m²/V/s)**

How to get Ohm's Law?

- assume free electrons in vacuum



$$F = ma = eE$$

$$a = \frac{dv}{dt} = \frac{F}{m} = \frac{eE}{m}$$

$$v = \frac{eE}{m} t \to \infty$$

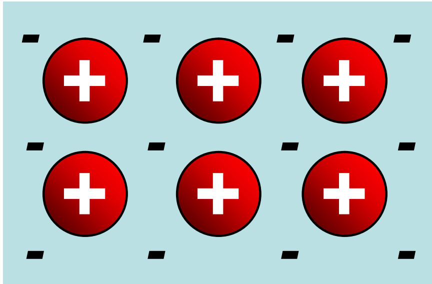
and

$$\sigma = ne\frac{v}{E} \to \infty$$

what is wrong?

The Drude Model 德鲁德模型

Free electron 'gas'



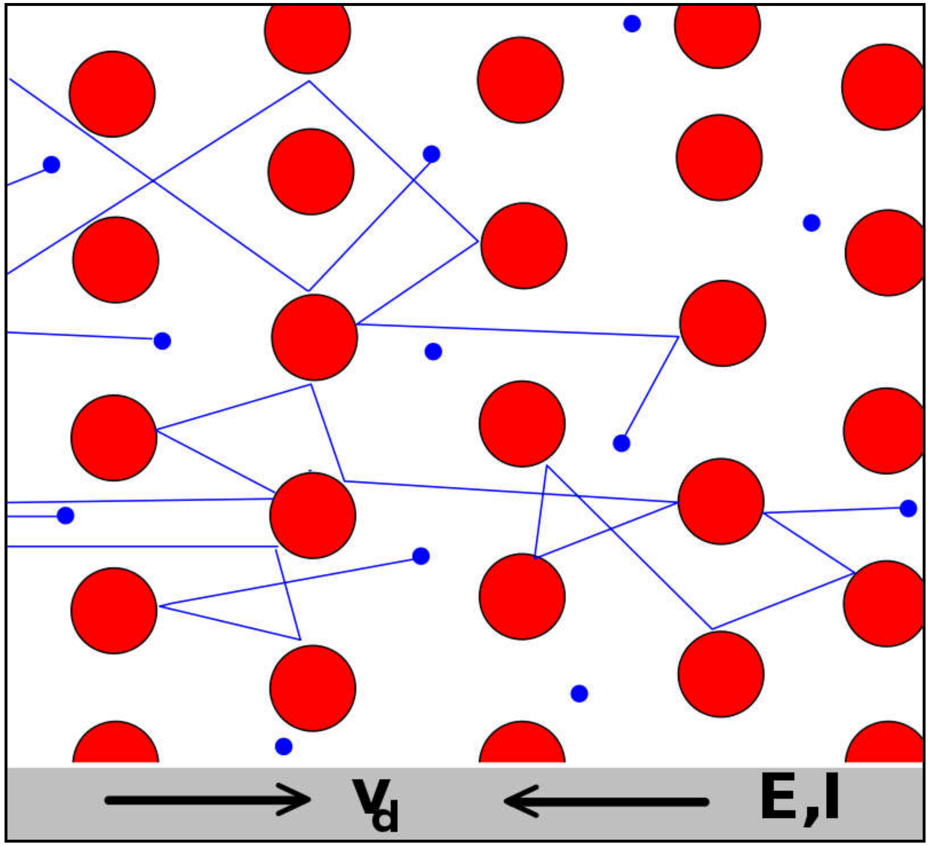
- Independent
  - electrons do not interact with each other
- Free
  - electrons do not interact with ions, except collision
- Collision (Origin of the resistance)
  - electrons are scattered by the ions instantaneously
- Relaxation time τ 弛豫时间
  - average time between two collisions
  - electron mean free path $l = v^*\tau$
- Maxwell–Boltzmann distribution
  - average kinetic energy

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$

P. Drude
1863–1906

The Drude Model 德鲁德模型

Free electron 'gas'



Resistance is caused by *collisions* of electrons with atoms

The Drude Model 德鲁德模型

Drude-Lorentz Model

$$F = m\frac{dv}{dt} + m\frac{v}{\tau} = eE(t)$$

τ - relaxation time (s) 弛豫时间

when *E* is constant, *v* is constant

$$v = eE\frac{\tau}{m}$$

$$\mu = \frac{v}{E} = e\frac{\tau}{m}$$

mobility

$$\sigma = ne\mu = ne^2\frac{\tau}{m}$$

conductivity

$$j = nev = \sigma E$$

Ohm's law

Successes of The Drude Model

- **Ohm's Law**

$$j = \sigma E$$

- **Electronic conductivity** σ

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T \qquad \tau = \frac{l}{v} \qquad \sigma = ne^2\frac{\tau}{m}$$

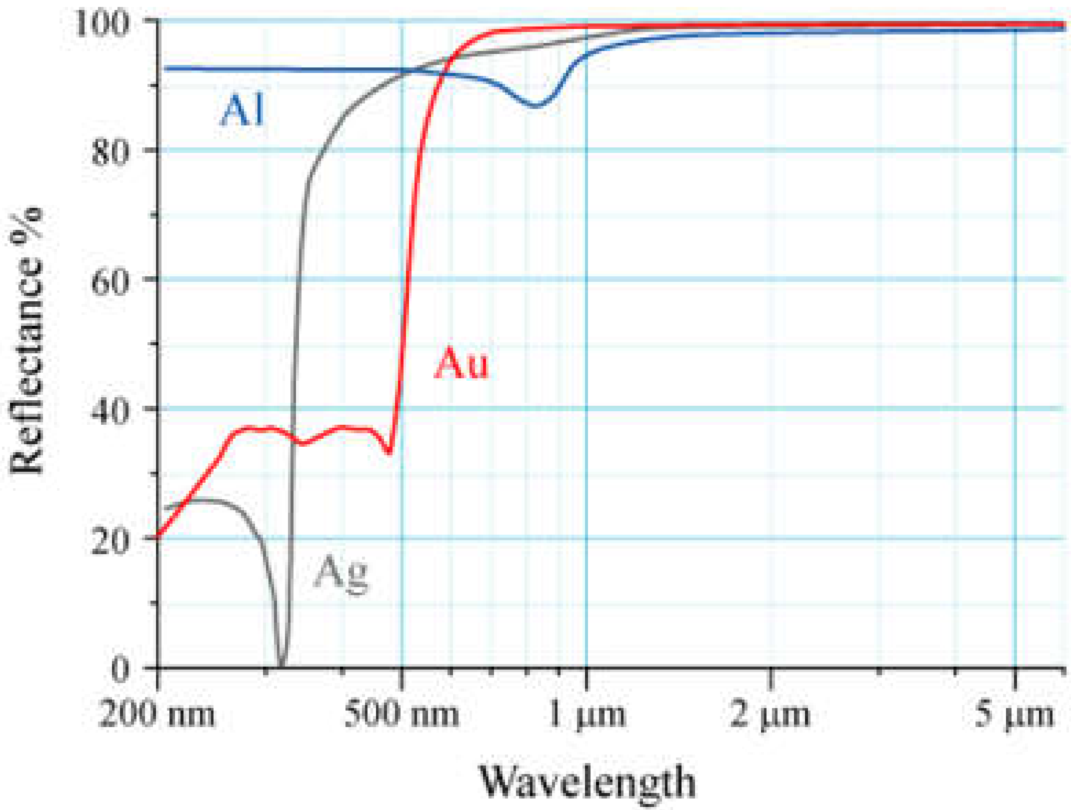
$v \sim 10^5$ m/s **$\tau \sim 10^{-14}$ s** **$\sigma \sim 10^7$ S/m**

***m* = electron mass 9.11×10^{-31} kg**
***k* = 1.38×10^{-23} J/K**
***e* = 1.6×10^{-19} C**
***T* = 300 K, room temperature**
***l* = mean free path 0.1~1 nm**
***n* = atomic density ~ 10^{29} #/m^3**

metals	conductivity (S/m) at 300 K
Ag	6.3×10^7
Cu	6.0×10^7
Al	3.5×10^7

Successes of The Drude Model

- **Optical Reflectivity of Metals**



mirror reflection

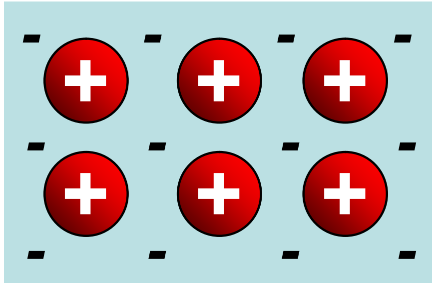
Failures of the Drude Model

It *cannot* explain

- **Electronic heat capacity**
- **Thermal conductivity**
- **Hall effect / Hall coefficient**
- **Insulators / Semiconductors**
- **...**

What was wrong?

The Drude Model 德鲁德模型



positive ions
+
electron cloud

Free electron 'gas'

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$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$

P. Drude
1863–1906

The Sommerfeld Model 索末菲模型

Free electron 'Fermi' gas

A. Sommerfeld
1868–1951

- Introduce quantum mechanics
- ~~Maxwell–Boltzmann distribution~~

→ Fermi–Dirac distribution

The Electron Wave Function

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\cdot\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

free electron

$$V(\mathbf{r}) = 0$$

→ $$\nabla^2\psi(\mathbf{r}) = -k^2\psi(\mathbf{r})$$

$$k^2 = \frac{2mE}{\hbar^2}$$

→ $$\psi(\mathbf{r}) = \sum_{\mathbf{k}} A_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r})$$

$$\int_V \psi^* \cdot \psi d\mathbf{r} = 1$$

→ **one solution is** $$\psi(\mathbf{r}) = \frac{1}{\sqrt{V}}\exp(i\mathbf{k}\cdot\mathbf{r})$$

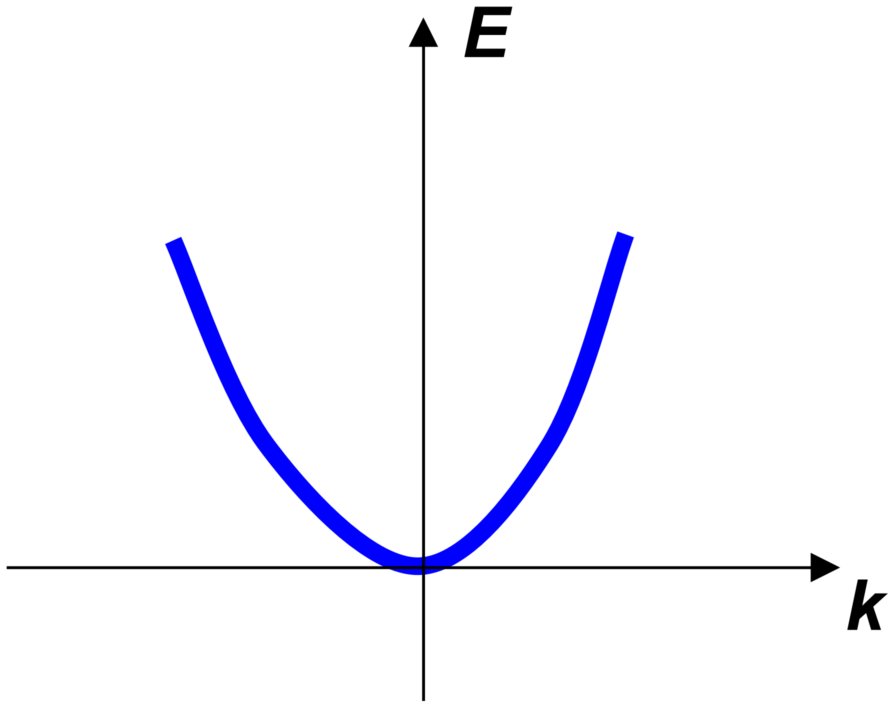
The Electron Wave Function

energy $$E(k) = \frac{\hbar^2 k^2}{2m}$$

velocity $$v = \frac{\hbar k}{m} = \frac{1}{\hbar}\frac{dE(k)}{dk}$$

momentum $$p = \hbar k$$

electron mass $$\frac{1}{m} = \frac{1}{\hbar^2}\frac{d^2E(k)}{dk^2}$$



E-k **diagram**
(energy dispersion curve)

Classical vs. Quantum

	Classical	Quantum	
velocity	v	wavenumber 波数 wavevector 波矢	$k = \frac{2\pi}{\lambda}$
momentum 动量	$p = mv$		$p = \hbar k$
energy 能量	$E = \frac{p^2}{2m}$		$E = \frac{\hbar^2 k^2}{2m}$

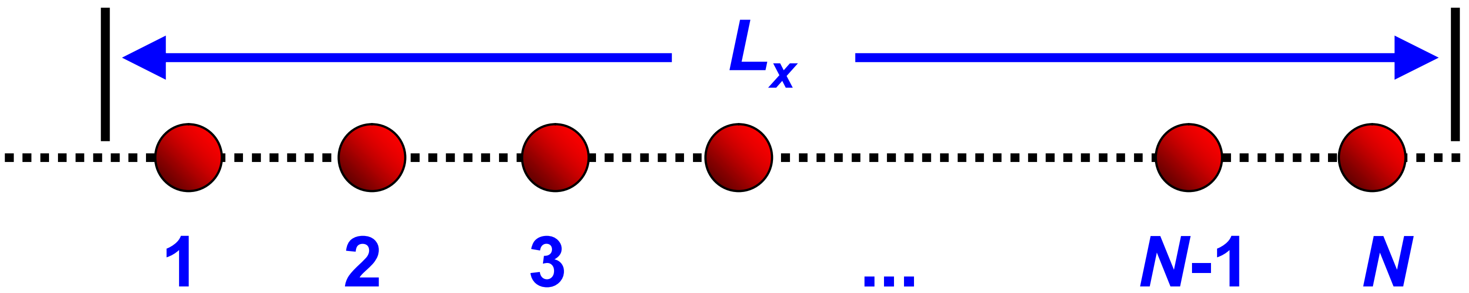
1D atomic chain

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \exp(i\mathbf{k} \cdot \mathbf{r})$$

→

1D atomic chain

$$\psi(x) = \frac{1}{\sqrt{L_x}} \exp(ik_x x)$$

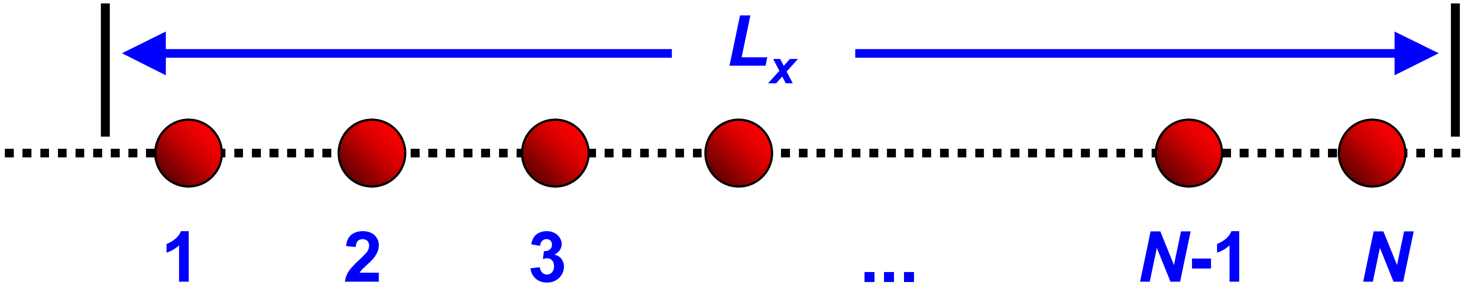


N **is large ~ 10^{23}**

Born-von Karman *periodic* boundary condition

$$\psi(x) = \psi(x + L_x) \rightarrow \exp(ik_x L_x) = 1$$

1D atomic chain



N **is large ~ 10^{23}**

Born-von Karman *periodic* boundary condition

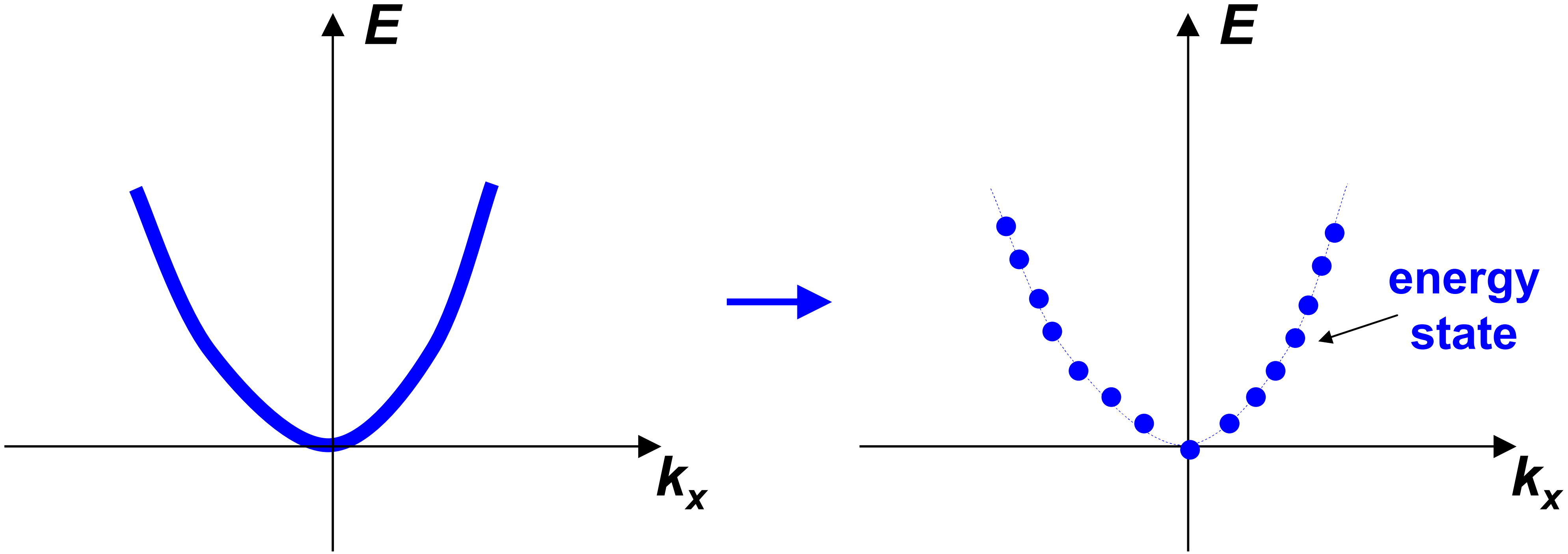
$$\psi(x) = \psi(x + L_x) \rightarrow \exp(ik_x L_x) = 1$$

$$\rightarrow \quad k_x = \frac{2\pi n_x}{L_x} \qquad n_x = 0, \pm 1, \pm 2, \ldots$$

k **is a *quantized* value**

1D atomic chain

$$k_x = \frac{2\pi n_x}{L_x} \qquad n_x = 0, \pm 1, \pm 2, \ldots$$



E-*k* diagram
(energy dispersion curve)

quantized energy
state / level

1D atomic chain

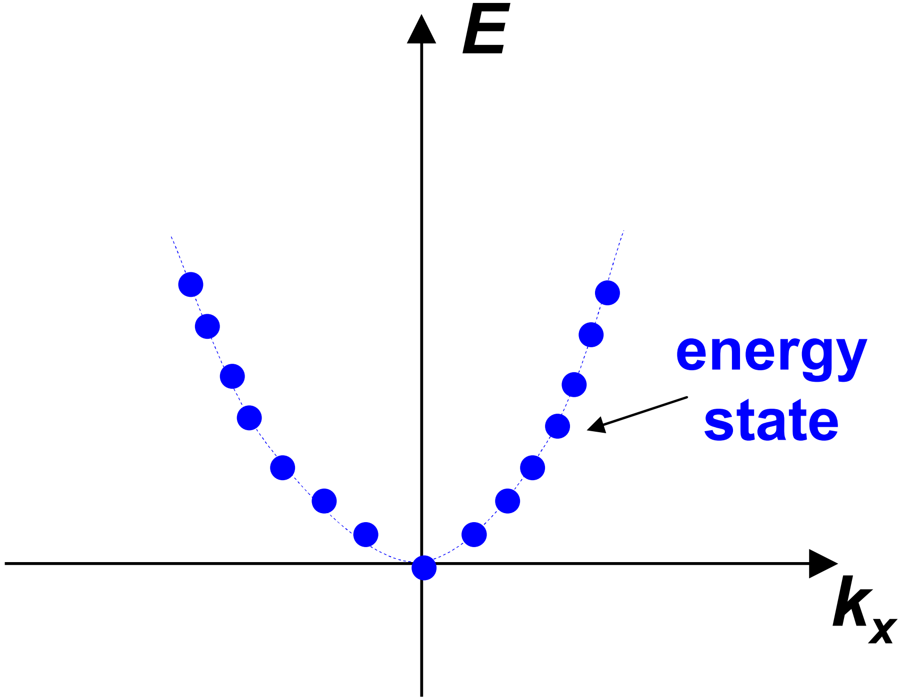
energy $$E(k) = \frac{\hbar^2 k^2}{2m}$$

$$k_x = \frac{2\pi n_x}{L_x}$$ $$n_x = 0, \pm 1, \pm 2, \ldots$$

velocity $$v = \frac{\hbar k}{m} = \frac{1}{\hbar}\frac{dE(k)}{dk}$$

momentum $$p = \hbar k$$

electron mass $$\frac{1}{m} = \frac{1}{\hbar^2}\frac{d^2 E(k)}{dk^2}$$



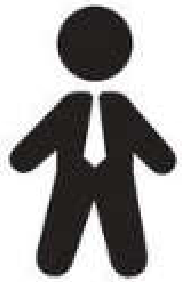
When *L* is large,
k and *E(k)* are *quasi-continuous*
(准连续)

quantized energy state / level

State vs. Electron

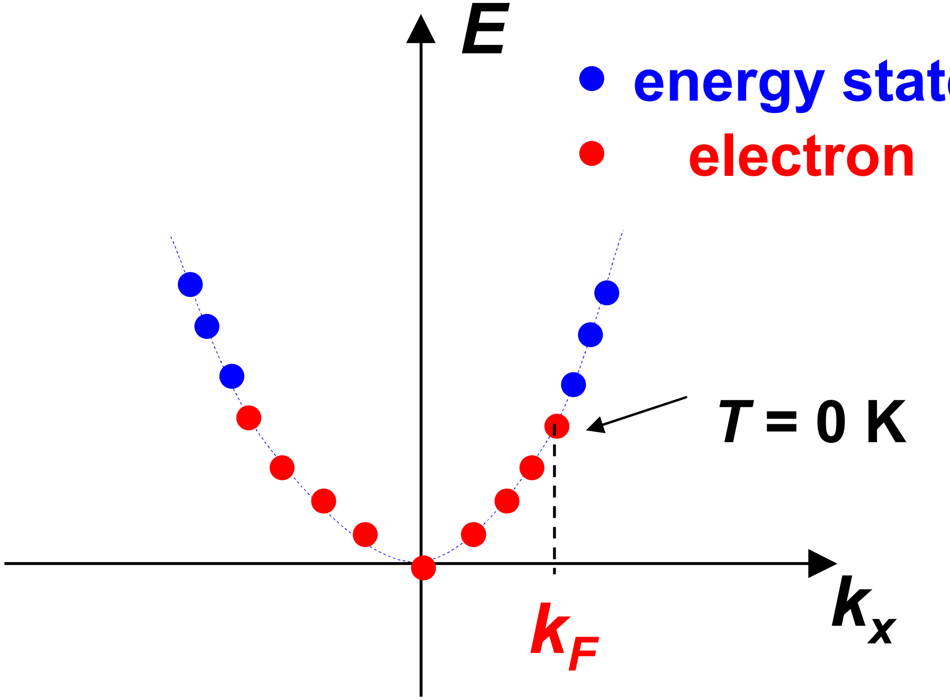
energy state / level / orbital
能态 / 能级 / 轨道

electron / phonon / ...
电子 / 声子 / ...



determined by space, lattice, environments, ...

1D atomic chain



$$k_x = \frac{2\pi n_x}{L_x} \qquad n_x = 0, \pm 1, \pm 2, \ldots$$

k_F - Fermi wavevector
highest *occupied* state at $T = 0$ K

Fermi-Dirac distribution:
spin up and down

$$-k_F < k_x < +k_F$$

$$N = 2 \cdot \frac{2k_F}{2\pi / L_x}$$

$$k_F = \frac{\pi}{2} \frac{N}{L_x}$$

N - total number of free electrons

N/L - free electron density

1D atomic chain

$$N = 2 \cdot \frac{2k_F}{2\pi / L_x}$$

1D atomic chain

$$k_F = \frac{\pi}{2}\frac{N}{L_x}$$

$$v_F = \frac{\hbar k_F}{m_e}$$

***k_F* - Fermi wavevector**
***E_F* - Fermi energy**
***v_F* - Fermi velocity**
***T_F* - Fermi temperature**

$$E_F = \frac{\hbar^2 k_F^2}{2m_e}$$

$$T_F = \frac{E_F}{k_B}$$

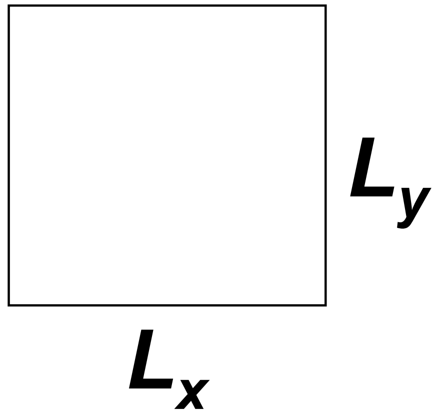
These values are determined by electron density N/L, not N or L

highest *occupied* state at *T* = 0 K

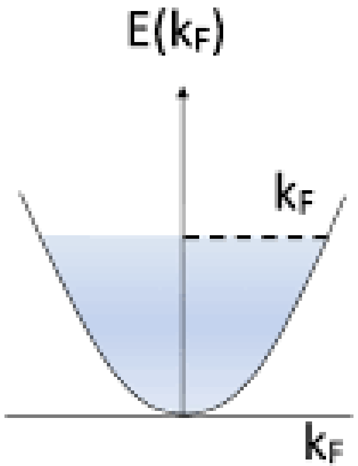
2D box

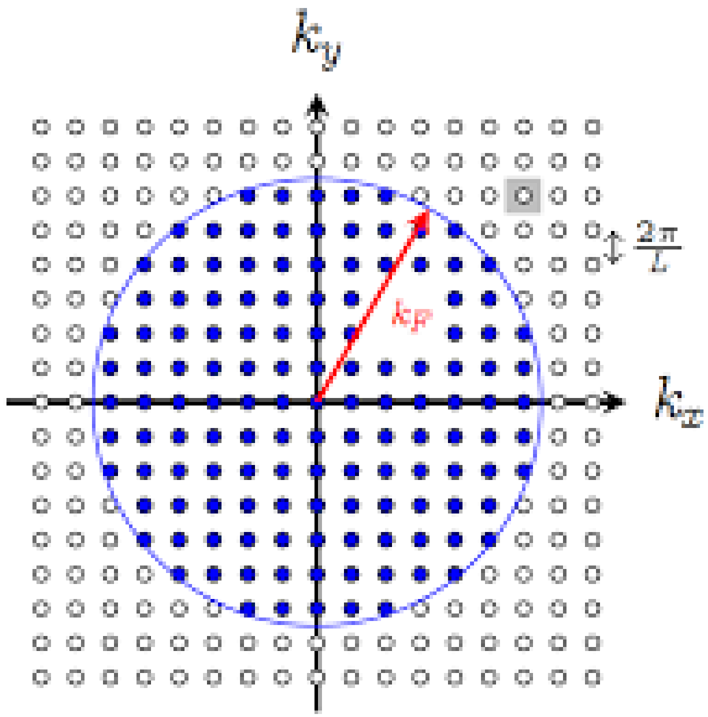
in a 2D box

$$k_x = \frac{2\pi n_x}{L_x}, k_y = \frac{2\pi n_y}{L_y}$$



$A = L_x L_y$





2D Fermi circle (费米圆)

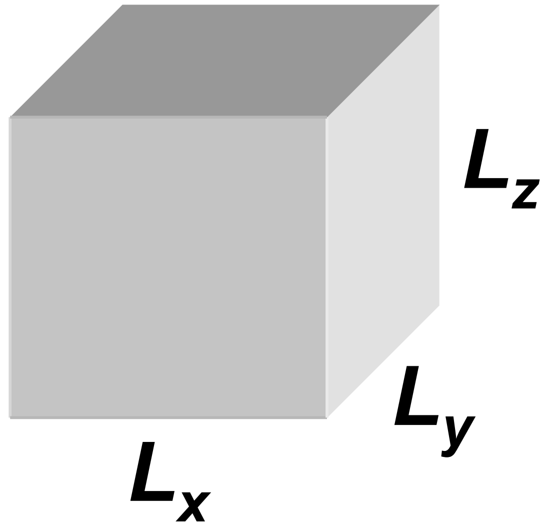
$$N = 2 \cdot \frac{\pi k_F^2}{\frac{2\pi}{L_x} \cdot \frac{2\pi}{L_y}}$$

$$= 2 \cdot \pi k_F^2 \cdot \frac{A}{(2\pi)^2}$$

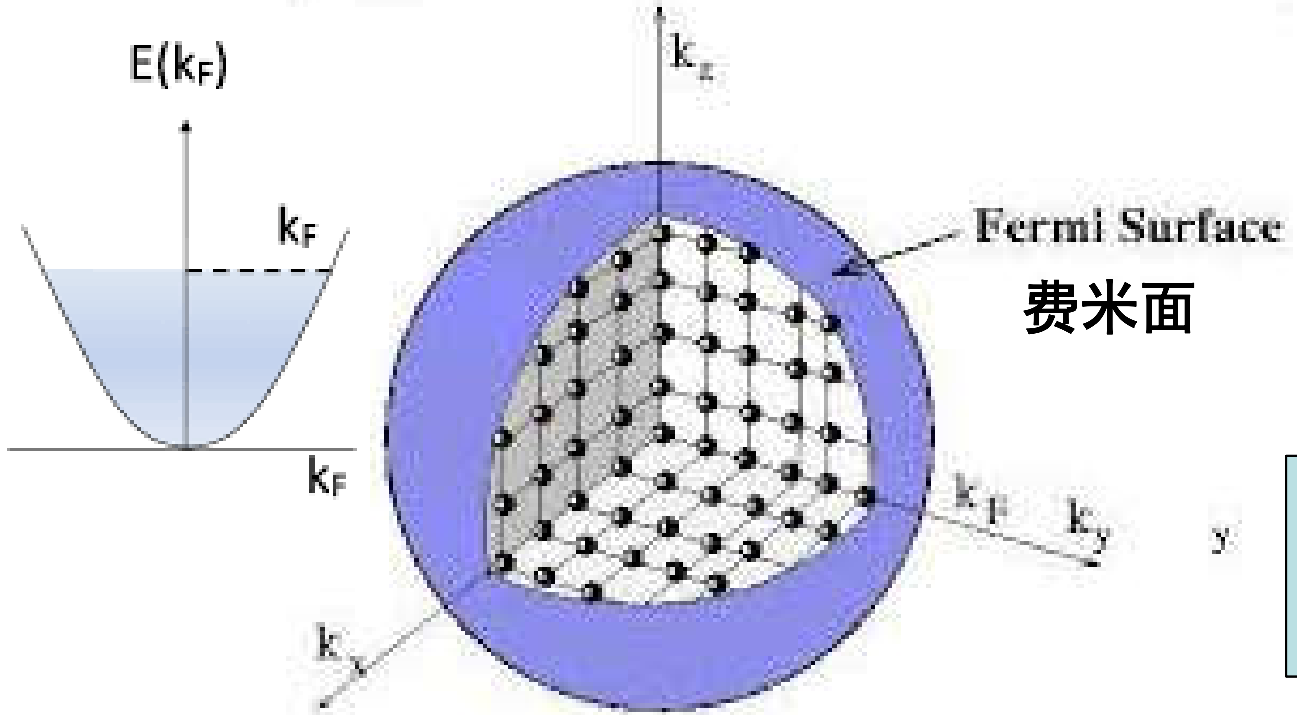
3D solid

in a 3D solid

$$k_x = \frac{2\pi n_x}{L_x}, k_y = \frac{2\pi n_y}{L_y}, k_z = \frac{2\pi n_z}{L_z}$$



$$V = L_x L_y L_z$$



$$N = 2 \cdot \frac{4\pi}{3} k_F^3 \cdot \frac{V}{(2\pi)^3}$$

3D Fermi sphere (费米球)

3D solid

$$N = 2 \cdot \frac{4\pi}{3} k_F^3 \cdot \frac{V}{(2\pi)^3}$$

in a 3D solid

***n* = *N*/*V* - free electron density**

k_F - Fermi wavevector
E_F - Fermi energy
v_F - Fermi velocity
T_F - Fermi temperature

$$k_F = (3\pi^2 n)^{1/3}$$

$$v_F = \frac{\hbar k_F}{m_e}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m_e}$$

$$T_F = \frac{E_F}{k_B}$$

These values are determined by density n, not N or V

highest *occupied* state at *T* = 0 K

Density of States (DOS) 态密度

$g(k)$

DOS - number of energy states/levels in *k* space

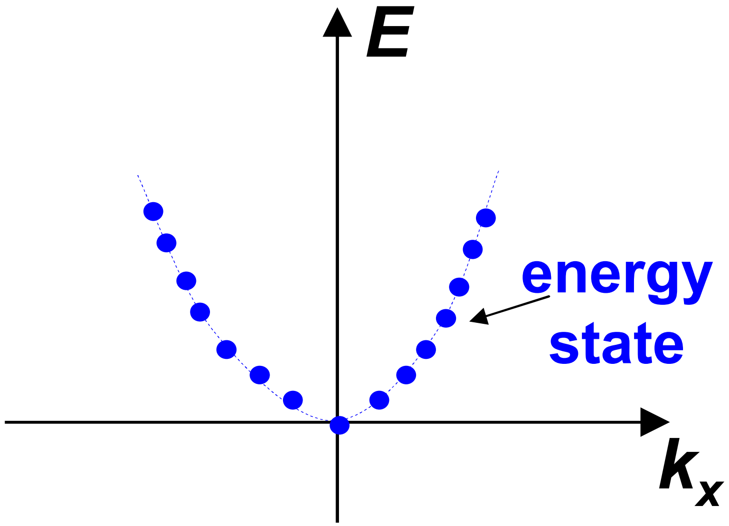
Ashcroft & Mermin, p.35

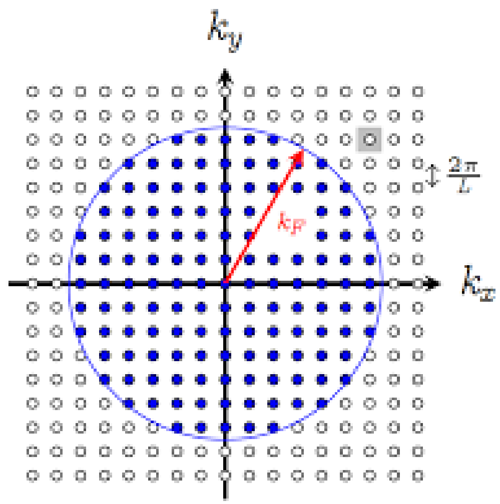
$$g(k) = 2\frac{L}{2\pi}$$

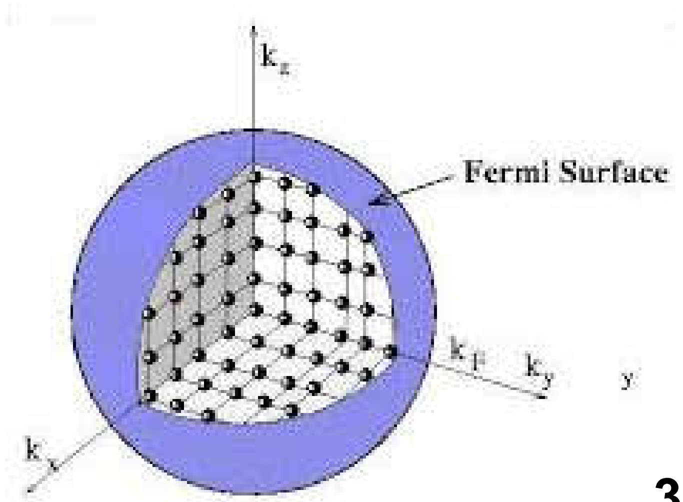
$$g(k) = 2\frac{A}{(2\pi)^2}$$

$$g(k) = 2\frac{V}{(2\pi)^3}$$

1D **2D** **3D**







Density of States (DOS) 态密度

$$g(E) = \frac{dn}{dE}$$

DOS - number of energy states/levels per unit energy in [*E*, *E*+*dE*], per unit volume

$$k = (3\pi^2 n)^{1/3}$$

$$E = \frac{\hbar^2 k^2}{2m_e}$$

→

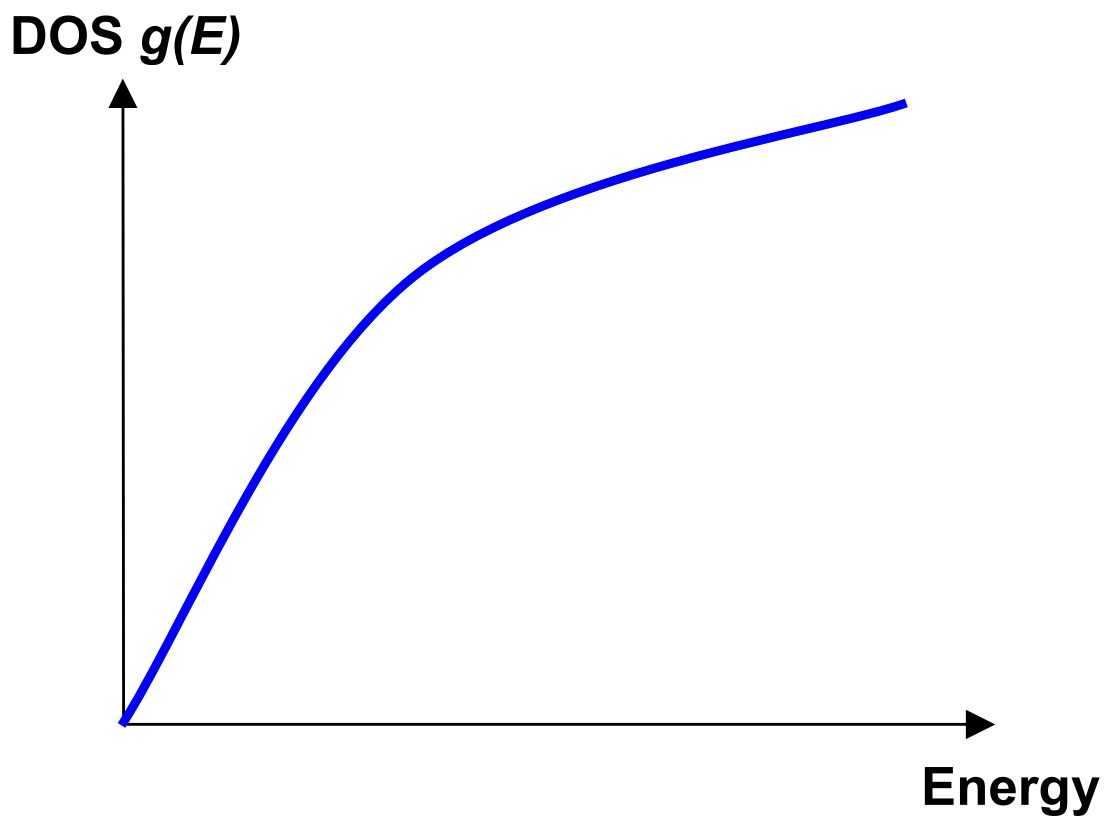
$$n = \frac{1}{3\pi^2}\left(\frac{2m_e}{\hbar^2}\right)^{3/2} E^{3/2}$$

→

$$g(E) = \frac{dn}{dE} = \frac{1}{2\pi^2}\left(\frac{2m_e}{\hbar^2}\right)^{3/2} E^{1/2}$$

Density of States (DOS) 态密度

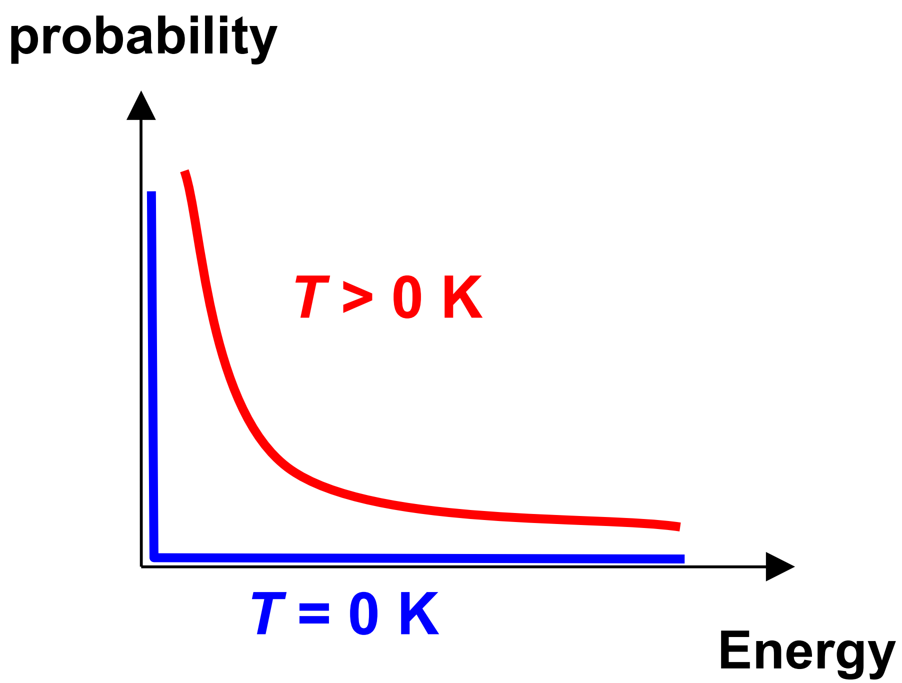
$$g(E) = \frac{dn}{dE} = \frac{1}{2\pi^2}\left(\frac{2m_e}{\hbar^2}\right)^{3/2} E^{1/2}$$



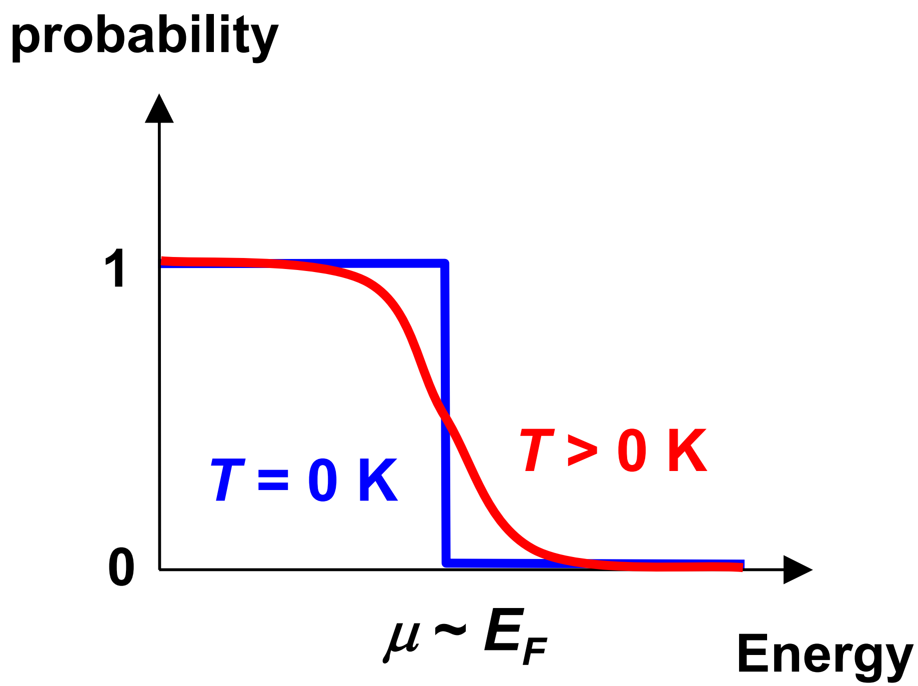
Q: How about in 1D and 2D?

Distribution of Free Electrons

Drude Model
Maxwell–Boltzmann



Sommerfeld Model
Fermi–Dirac



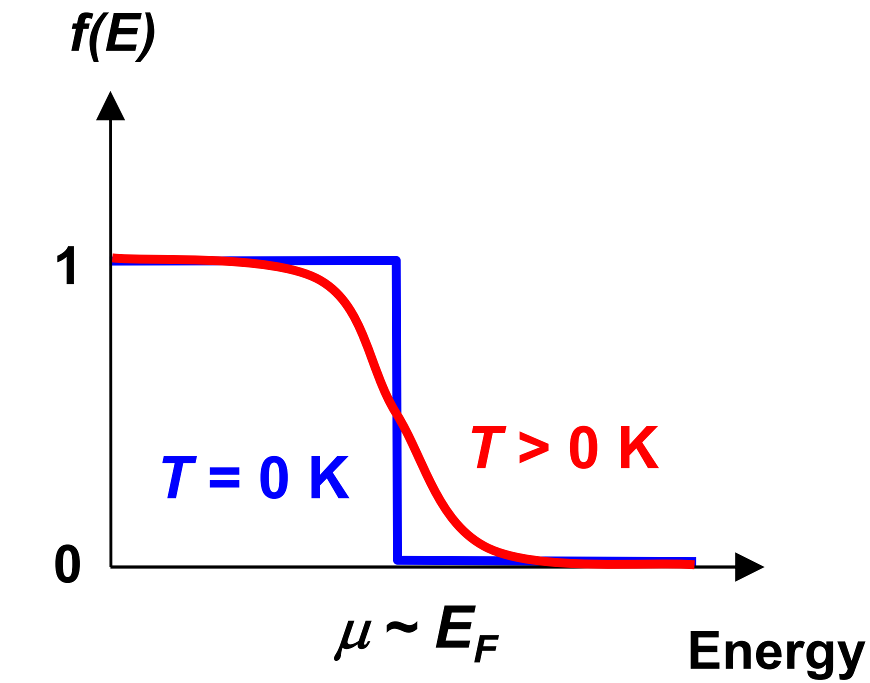
Electrons occupy higher levels of energy, even at 0 K

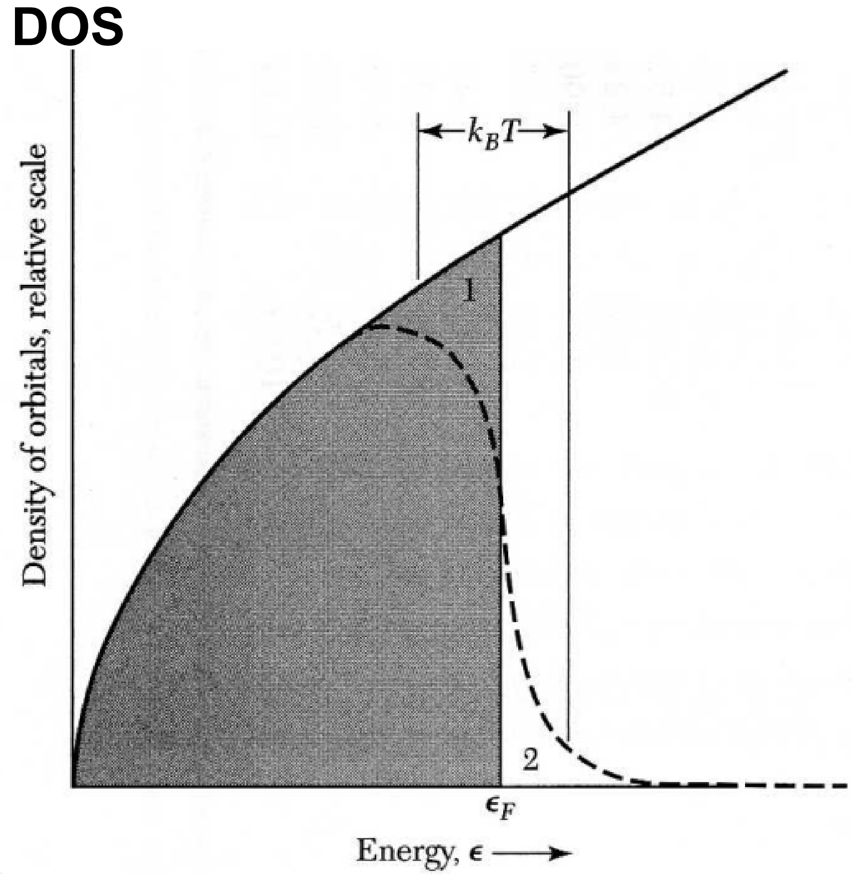
Density of Electrons

Density of electrons = DOS * probability

$$f(E)g(E)$$

$$f(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1}$$





When T > 0 K, some electrons are excited to higher states (from 1 to 2)

State vs. Electron

energy state / level / orbital
能态 / 能级 / 轨道

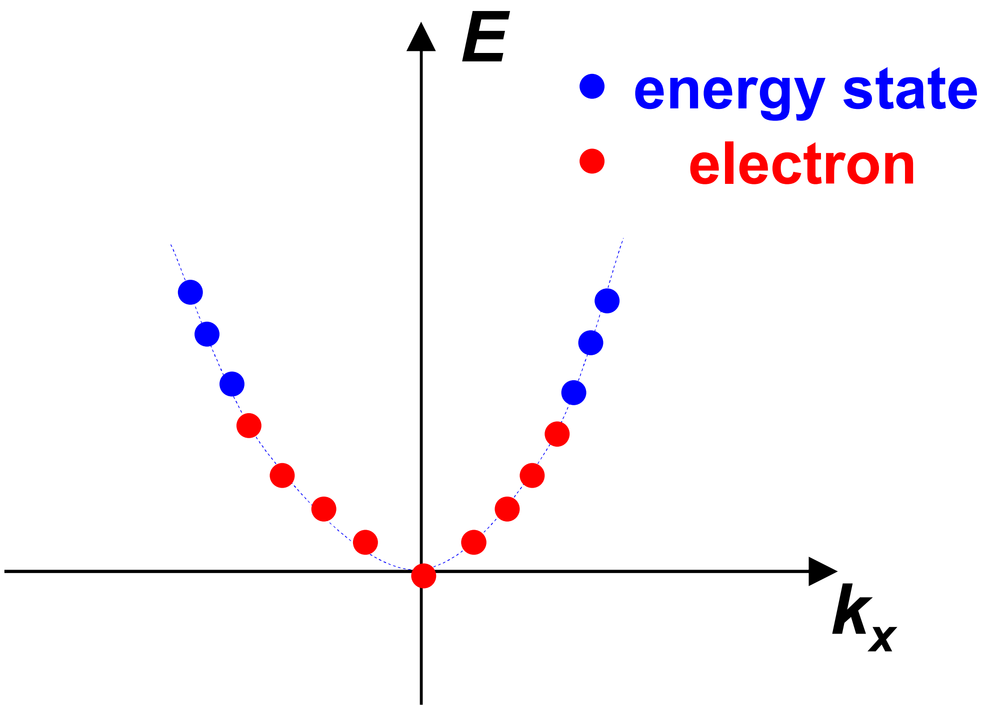
electron / phonon / ...
电子 / 声子 / ...

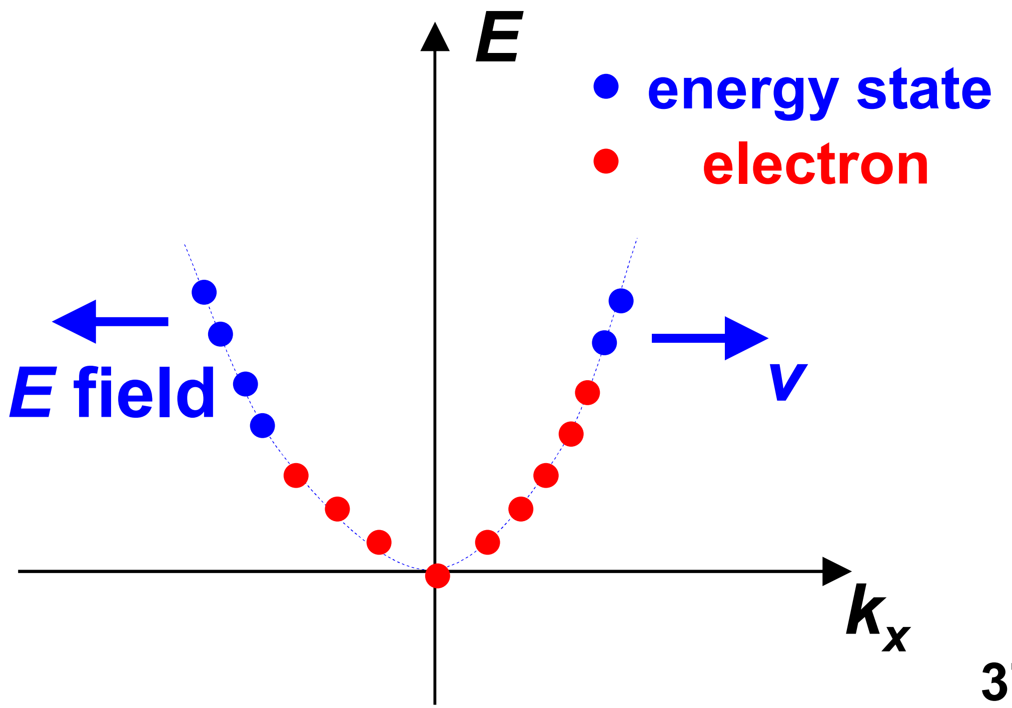
determined by space, lattice, environments, ...

Electrons in an Electric Field *E*

momentum 动量 $p = mv = \hbar k$

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = \hbar\frac{d\mathbf{k}}{dt} = -e\mathbf{E} \rightarrow \delta\mathbf{k} = -\frac{e\mathbf{E}}{\hbar}t$$





Electrons in an Electric Field *E*

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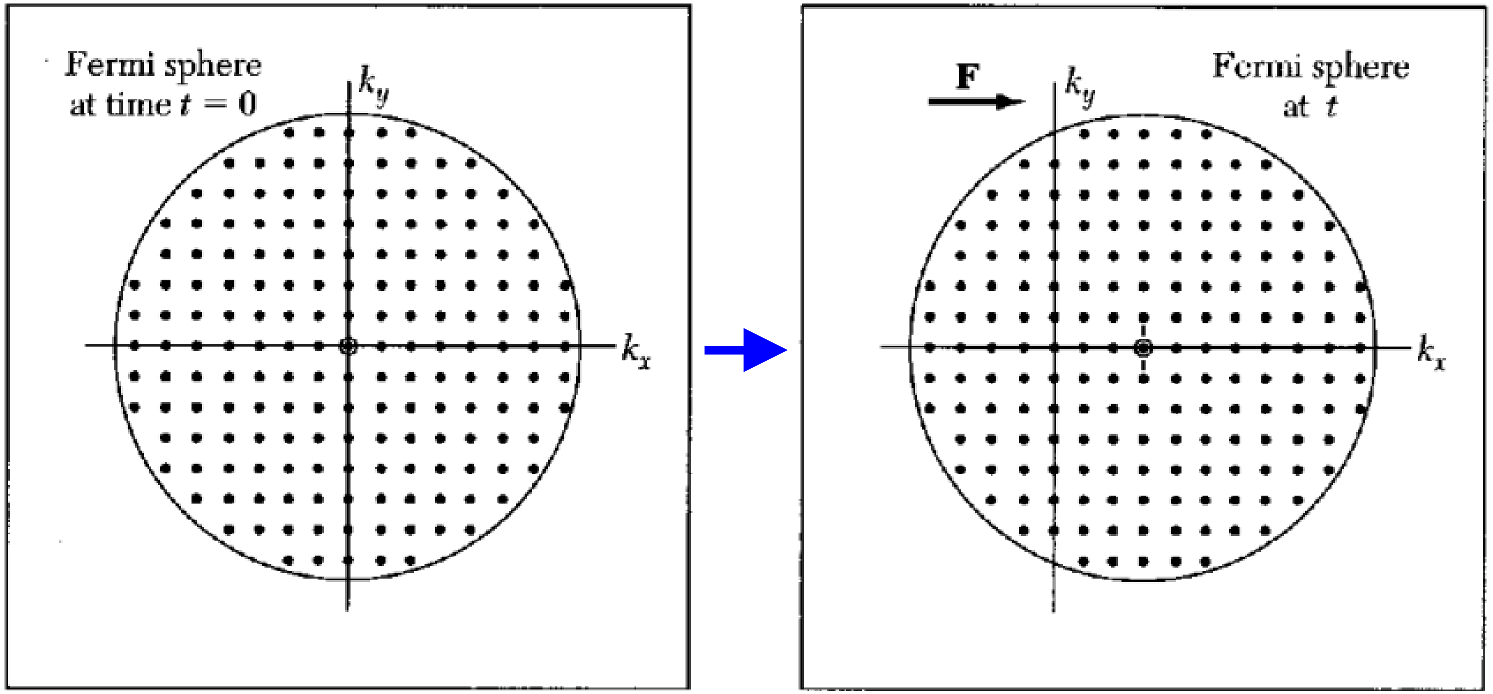


Electrons in an Electric Field *E*

momentum 动量 $p = mv = \hbar k$

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = \hbar\frac{d\mathbf{k}}{dt} = -e\mathbf{E} \rightarrow \delta\mathbf{k} = -\frac{e\mathbf{E}}{\hbar}t$$

collision time $t = \tau$, the displacement δk is steady

$$\delta\mathbf{k} = -\frac{e\mathbf{E}}{\hbar}\tau \rightarrow v = \frac{\hbar\delta\mathbf{k}}{m} = -\frac{e\mathbf{E}}{m}\tau$$

$$\rightarrow \mathbf{j} = -ne\mathbf{v} = \frac{ne^2\tau}{m}\mathbf{E} = \sigma\mathbf{E}$$

Ohm's law

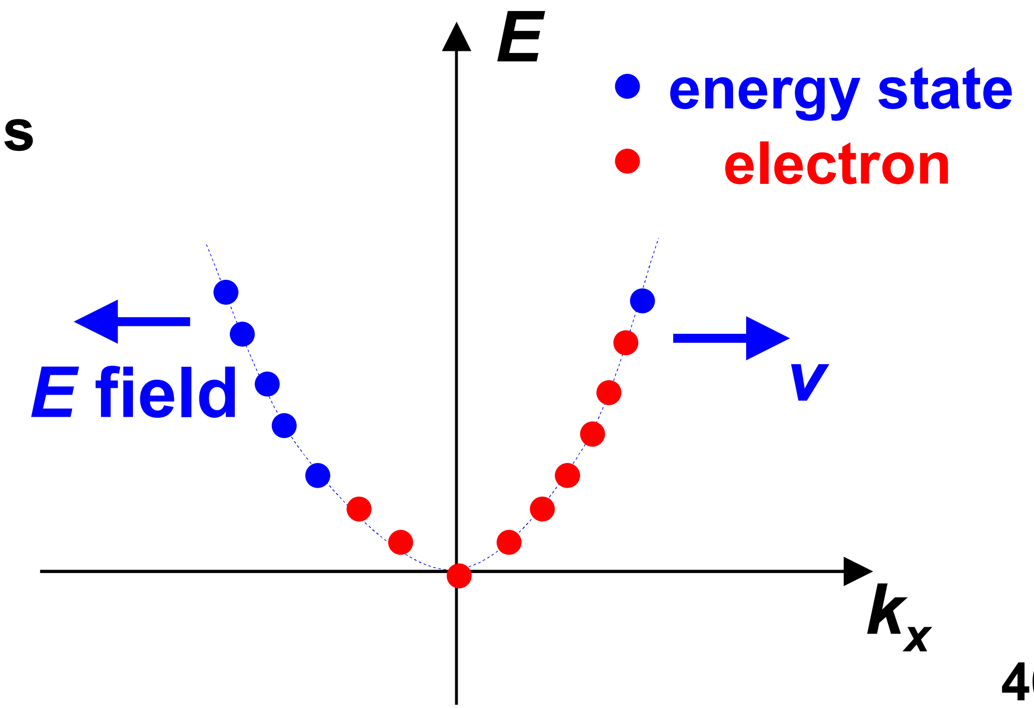
Electron Conductivity - Revisit

- **Electrons are in different energy states, therefore have different velocities and energies. Under *E* field, there are more electrons moving in the opposite direction.**

- **Mobility μ and relaxation time τ are average values for all the free electrons**

- **only σ is meaningful for metals**

$$\sigma = ne\mu = \frac{ne^2\tau}{m}$$



Success of The Sommerfeld Model

- **Ohm's Law**
- **Electronic conductivity** σ
- **Thermal conductivity of electrons**
- **Electronic heat capacity**

Failures of The Sommerfeld Model

It *cannot* explain

- Electronic / Thermal properties of some other metals
- Hall effect / Hall coefficient
- Insulators / Semiconductors
- ...

The Free Electron Models

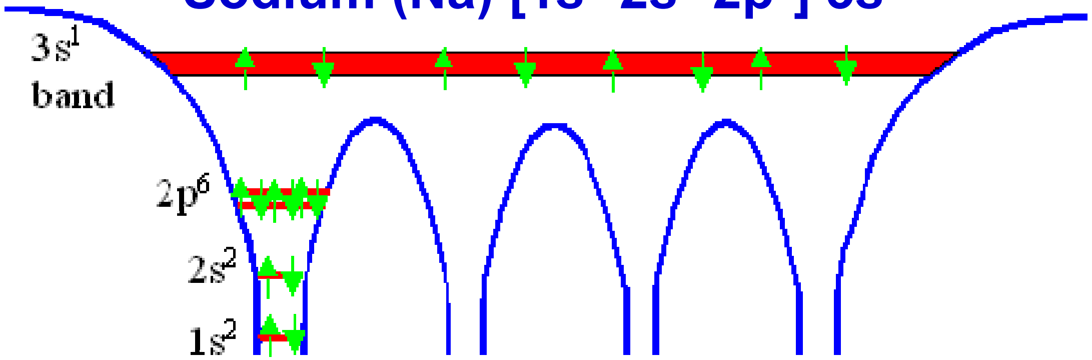
- The Drude Model: 1900s
- The Sommerfeld Model: 1920s

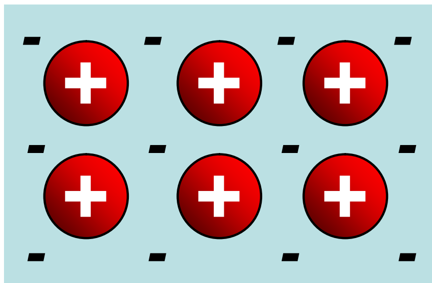
- What are missing?
  - Material and atom structures
  - Potentials of positive ions
  - Localized electrons
  - ...

P. Drude
1863–1906

A. Sommerfeld
1868–1951

Sodium (Na) [$1s^2$ $2s^2$ $2p^6$] $3s^1$





positive ions
+
electron cloud

Thank you for your attention